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PRESSURE SIGNALS IN RANDOM LINEARLY ELASTIC RODS

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Abstract—The dynamic response of a linearly elastic rod with material properties which are described by random functions of position along the rod is studied. A general formulation to be satisfied by the ensemble averaged field quantities is derived. This formulation is considered in some detail for those situations in which the magnitude of the random variations is vanishingly small and those in which the length scale on which these variations are observable is small compared to the length scale on which the variations in the averaged field quantities are observable. The infinite rod dispersion spectrum is obtained for the former situation. A dynamic "effective modulus" theory is obtained for the latter situation. The physical implications of the results are discussed.

INTRODUCTION

THE use of the formalism of stochastic processes to predict the "effective moduli" for solid media that possess an irregular substructure stems from the works of Brown, Beran and Lomakin [1–3]. Since that time several authors have exploited this idea with success [4–8]. The justification behind using the results of an analysis of a stochastic process to make a prediction of the results of an associated deterministic process rests on an ergodic hypothesis. As applied to the "effective modulus" prediction problem, ergodicity allows the equation of statistical (i.e. ensemble) averages with spatial averages. The claim to ergodicity is based on the fact that the desired effective moduli are ratios of two statistically homogeneous fields. (For example, a stress field and a strain field.) Statistical homogeneity refers to the invariance of certain statistical averages with a change in position.

More recently, Beran and McCoy [9, 10] borrowed an idea long used in related fields that enables one to make use of the idea of a random medium in analyzing an associated deterministic medium in those situations in which the field quantities of interest are not statistically homogeneous. To do so requires, in the present context, the identification of two length scales in describing the solid. On one scale, the inner scale, defined by the substructure one is unable to perceive the statistical inhomogeneities. That is, the variations in statistical averages occur over distances that appear unboundedly large when viewed on this scale. The outer scale, on the other hand, is defined by variations in these statistical averages. The variations in the substructure are too rapid to be discerned on this outer scale. The presence of two length scales allows one to once again make recourse to an ergodic hypothesis and equate a statistical average to a spatial average over a region that appears unboundedly large when viewed on the inner scale. This same region appears as a point on the outer scale. We note that it is this same condition, the presence of two length scales, that gives the concept of an "effective modulus" theory physical meaning. As the distinction between the two length scales vanishes so does the justification for invoking an ergodic hypothesis. Of course, in such a situation it is difficult to assign any physical significance to a spatial average. A statistical average will, however, still have meaning (see below). In this sense, one might say that the statistical approach is a more general approach.

In Ref. [10], equations are developed that govern the statistically averaged stress, strain and displacement fields in a three dimensional linearly elastic solid subjected to a static forcing. These equations are in the form of a non-local elasticity theory in that the averaged stress at a point depends on the averaged strain at all points in the solid. Introducing l, to denote a characteristic dimension of the substructure and L, to denote a characteristic dimension for the variations in the averaged field quantities, these equations were investigated in the limit as l/L becomes vanishingly small. The result is that, to a zeroth order approximation, the non-local nature of the equations vanishes and one obtains an "effective modulus" theory as one expects. A first order corrected theory was obtained and shown to have the form of Toupin's gradient theory [11]. However, by considering the case of a weakly inhomogeneous medium, it was shown that the sign of one of the higher order material parameters was such as to violate a positive definiteness requirement on the strain energy density of Toupin's theory. This point was investigated in a subsequent paper [12].

We turn now to a dynamical theory. Several interesting questions arise when inertia effects are to be included. What is meant by an "effective modulus" theory? Do we simply replace the material parameters and the mass density in the classical theory with their effective counterparts? Do we need an effective mass density or is it valid to use an average mass density? Weighing the specimen will give an average value for the mass density. Will the dynamical effective moduli be the same as the associated statical effective moduli? It is to be expected that a dynamical effective theory will be a long wavelength, low frequency theory. Will a solid with a disordered substructure admit of averaged solutions that are characterized by long wavelength but high frequency? These would be analagous to the optical branch solutions that exist in solids with an ordered microstructure. It is the purpose of the present paper to consider such questions as well as to consider the extension of an "effective modulus" theory. Attention is restricted to a one-dimensional theory and we consider the propagation of pressure signals through a random linearly elastic rod. The one-dimensional problem is considered because it simplifies the algebraic manipulations; it represents a physically meaningful problem; and the results should lend themselves to laboratory verification. Prior work [13–15] on the dynamic response of three dimensional linear elastic solids with a disordered substructure considered the effect of the substructure on the nondispersive waves that exist in the solid without the substructure in the limit of the substructure vanishing (i.e. a weakly inhomogeneous solid).

By a random linearly elastic rod is meant an ensemble of rods, each element of which is described by a Young's modulus, $E(x)$, and a mass density, $\rho(x)$, that are given by irregular functions of position measured along the rod. The average of these functions taken over the ensemble are taken to be smooth functions of position. In all of the detailed calculations to be presented these averages are taken to be independent of position. (We restrict ourselves to statistically homogeneous random linearly elastic rods.) The space–time history of a pressure signal as it exists in each rod will differ from one another in an irregular manner. We are interested in the averaged space–time history of a pressure signal where the average is taken over the ensemble. One might gain an understanding of the nature of the information in this average by considering two extreme examples. The rods of interest are taken to be infinite in extent and to be such that $E(x)$ and $\rho(x)$ are piecewise constant. The change from one constant value to another occurs over a distance that approaches one that is vanishingly small. Each rod is to be excited by a point source which emits a tone burst of frequency ω and duration T. In one of the two extremes, the ensemble is such that the burst gives rise to waves in the rods with lengths, L, that are very much longer than the largest distances over which $E(x)$ and $\rho(x)$ are constant l. That is, $L/l \gg 1$. In the second extreme the ensemble is such that $L/l \ll 1$. In Figs. 1 and 2 are exhibited schematic representations

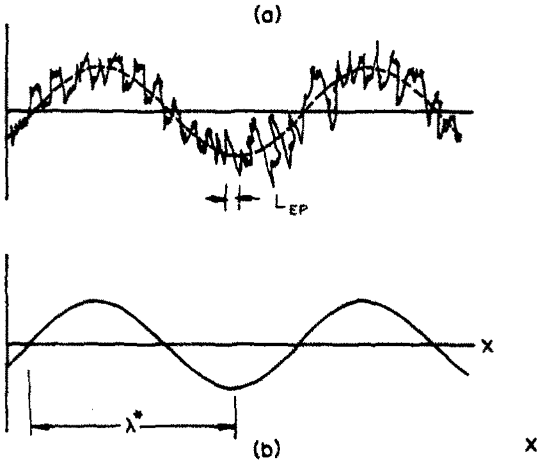

FIG. 1. Extreme in which lengths of waves generated by burst $[0(\lambda^*)]$ are large compared to lengths of constant E and $\rho[0(l_{E,\rho})]$. Upper plot illustrates schematically a portion of the spatial distribution that results in an element of ensemble after some time. Lower plot illustrates average taken over all elements of ensemble. Same result obtained from spatial average of above over region that is large on $l_{E,\rho}$ scale but small on λ^* scale.

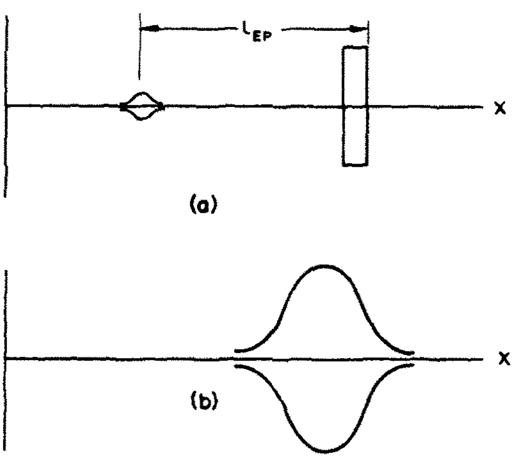

FIG. 2. Extreme in which lengths of waves generated by burst $[0(\lambda^*)]$ are small compared to constant E and $\rho[0(l_{E,\rho})]$. Upper plot illustrates schematically the succession of smooth signals that arise due to interaction of signal and interface. Lower plot illustrates average taken over all elements of ensemble. Only the envelopes are shown since the detailed variations are too rapid to be viewed on this scale.

of the spatial distributions of the resulting signals that one might expect to exist in a single element of each of the two ensembles at some later instant of time. In the first extreme ($L/l \gg 1$), the signals generated by the multiple reflections and transmissions occurring at each interface are spatially displaced relative to one another by a small amount. The combination of all of these overlapping signals is the complicated interference pattern shown. In the second extreme ($L/l \ll 1$) the first large signal is the result of a transmission at each section. It is an undistorted replica of the signal originally generated although its amplitude has been decreased due to the energy that is lost to this signal by virtue of a reflection at each intersection. The second signal is a superposition of signals resulting from a reflection at two sections and transmission at the others. It is shown to be of much lower amplitude from the first signal which would be the case for a weakly inhomogenous rod. Signals resulting from reflections at more than two sections combine to form a succession of distant distorted signals. They are not shown. (We note that on the scale shown the variations within the tone burst are too rapid to be discerned.) Also shown in the figures are schematic representations of the ensemble averaged signals that one might expect in the two examples. In the first extreme the irregular variations that result due to the small differences in the locations of the many transmitted and reflected signals are smoothed out. This same result is obtained by a spatial averaging of the signal measured in an individual rod. The spatial average is over a region that is large compared to the irregular variations of unaveraged signal, l, but small compared to the smooth variations of the averaged signal, L. This is an example of a case in which we are justified in invoking an ergodic hypothesis. In the second example there is no resemblance between the averaged signal and the signal that exists in any one rod. This is a non-ergodic process. We note, however, that there is still information in the ensemble averaged signal. Looking to the first signal, it seems clear that the area of the signal contains information on the probable amount of energy lost to the first signal by reflections at the interfaces. The spatial distribution of the signal contains information on the probable location of the undistorted first signal that occurs in a single element of the ensemble.

The format of the paper is as follows. A derivation of the equations governing the ensemble averaged response is given in the next section. Several simplifications of these equations that result from certain physical approximations are considered in section three. In particular we consider a weakly inhomogeneous limit and a low frequency, long wavelength limit. The latter gives an "effective modulus" theory. In section four the weakly inhomogeneous rod is considered in more detail. The dispersion spectrum for a statistically homogeneous ensemble of infinitely long rods is obtained. The physical information contained in this spectrum is discussed. In the last section the results are summarized and the effect of introducing a boundary is speculated upon.

DERIVATION OF EQUATIONS GOVERNING MEAN FIELD

Since an excellent discussion of the history and validity of using the following derivation procedure in propagation problems is available [16], we shall ignore all such questions here and simply apply it to the problem at hand. It is more instructive to first present a derivation using an operator notation and only introduce the specifics into the final result. In the operator notation the fundamental problem is given by the equation

$$Lu = g, \tag{1}$$

where L denotes a linear stochastic operator, u denotes here, the desired solution variable, also stochastic and g denotes a forcing taken to be deterministic. We are interested in developing an equation on $\langle u \rangle$, where $\langle\ \rangle$ denotes an ensemble average.

Averaging equation (1) gives

$$\langle L \rangle \langle u \rangle + \langle L'u' \rangle = g. \tag{2}$$

A prime is used to denote the fluctuating part of the quantity to which it is attached about its mean value. Subtracting equation (2) from (1) gives

$$\langle L \rangle u' + (I-P)L'u' = -L'\langle u \rangle, \tag{3}$$

where I denotes the identity operator and P (for projection) denotes the operation of taking an ensemble average. Equation (3) is now to be viewed as an equation on u' in which the r.h.s. plays the role of a known forcing. Upon solving it we obtain u' in terms of $\langle u \rangle$ and L which is then substituted into equation (2) to give the desired equation on $\langle u \rangle$.

In solving equation (3) we make use of a Liousville–Neumann series, i.e.

$$u' = -\sum_{n=0}^{\infty} [-\langle L \rangle^{-1}(I-P)L']^n \langle L \rangle^{-1} \langle u \rangle. \tag{4}$$

Here, the notation $\langle L \rangle^{-1}$ denotes the inverse of the operator $\langle L \rangle$. Thus the only inverse we actually construct is that of a deterministic operator. Equation (4) into (2) gives an equation that is frequently termed the Dyson equation.

$$\left[\langle L \rangle - \sum_{n=0}^{\infty} (-1)^n PL'[\langle L \rangle^{-1}(I-P)L']^n \langle L \rangle^{-1} L'\right] \langle u \rangle = g. \tag{5}$$

This equation may be written as

$$[\langle L \rangle - M]\langle u \rangle = g, \tag{5a}$$

where the operator M, termed the mass operator, is obtained by comparison. Notice that M is known only in the form of an infinite series of component operators.

For the application of interest, the governing equations are given by elementary rod theory and they may be written, using a matrix notation, in the following form:

$$\begin{pmatrix} \partial_x & -\partial_t & 0 & 0 \\ 1 & 0 & -E(x) & 0 \\ 0 & 1 & 0 & -\rho(x) \\ 0 & 0 & \partial_t & -\partial_x \end{pmatrix} \begin{Bmatrix} \tau(x,t) \\ p(x,t) \\ \varepsilon(x,t) \\ v(x,t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{6}$$

The stress, linear momentum, strain and velocity fields are represented by τ, p, ε and v, respectively; ∂_x and ∂_t denote partial differentiation with respect to distance along the rod, x, and time, t. Young's modulus is represented by $E(x)$ and the mass density by $\rho(x)$. They are to be random functions of position. We shall carry out detailed calculations only for an infinite statistically homogeneous rod. For the infinite rod, in order to keep our mathematics valid it is necessary to restrict the variations in $E(x)$ and $\rho(x)$ with position to some bounded region. Once the solution is obtained the limits of this bounded region may then be increased without bound. It is in this limit that $E(x)$ and $\rho(x)$ approach statistically homogeneous functions.

The displacement field, u,† can be obtained by integrating the solutions for ε and v according to

$$\begin{aligned} \varepsilon(x, t) &= \partial_x u(x, t) \\ v(x, t) &= \partial_t u(x, t). \end{aligned} \tag{7}$$

In addition to the field equations (6), it is necessary to specify auxiliary conditions to complete the formulation. One might think of the problem of a wave train originating at $x = -\infty$ to be incident on the domain over which $E(x)$ and $\rho(x)$varies. The question of interest applies to what happens to this wave train. A second problem might be the initial value problem in which the strain and displacement fields are specified at an instant of time, determine the subsequent time histories.

Comparison of equations (1) and (6) allows identification of L, u and g. We note

$$\langle L\rangle = \begin{pmatrix} \partial_x & -\partial_t & 0 & 0 \\ 1 & 0 & -\langle E\rangle & 0 \\ 0 & 1 & 0 & -\langle\rho\rangle \\ 0 & 0 & \partial_t & -\partial_x \end{pmatrix}, \tag{8}$$

and

$$L' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & E'(x) & 0 \\ 0 & 0 & 0 & \rho'(x) \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

The mean operator is a differential operator with constant coefficients (due to the statistical homogeneity of E and ρ); the fluctuating part of the operator is algebraic.

The inverse of $\langle L\rangle$ will be an integro-differential operator which we can define by introducing a Green's function matrix

$$\begin{pmatrix} G_{\tau 1} & G_{\tau 2} & G_{\tau 3} & G_{\tau 4} \\ G_{p1} & G_{p2} & G_{p3} & G_{p4} \\ G_{\varepsilon 1} & G_{\varepsilon 2} & G_{\varepsilon 3} & G_{\varepsilon 4} \\ G_{v1} & G_{v2} & G_{v3} & G_{v4} \end{pmatrix}. \tag{10}$$

The first column elements satisfy the field equations

$$\begin{pmatrix} \partial_x & -\partial_t & 0 & 0 \\ 1 & 0 & \langle E\rangle & 0 \\ 0 & 1 & 0 & \langle\rho\rangle \\ 0 & 0 & \partial_t & -\partial_\lambda \end{pmatrix} \begin{Bmatrix} G_{\tau 1}(x, t; x_1 t_1) \\ G_{p1} \\ G_{\varepsilon 1} \\ G_{v1} \end{Bmatrix} = \begin{Bmatrix} \delta(x-x_1)\delta(t-t_1) \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{11}$$

† A change in notation is to be noted. No confusion need result since the proper interpretation is always evident from the context.

where δ denotes the Dirac function, together with necessary auxiliary conditions. The second, third and fourth column elements satisfy similar problems with the singularity forcing in the second, third and fourth row, respectively. If we introduce the Green's function for the one-dimensional wave operator, i.e. the solution of the equation

$$(\partial_x^2 - C^{-2}\partial_t^2)G(x_1 t; x_1, t_1) = \delta(x-x_1)\delta(t-t_1), \tag{12}$$

where

$$c^2 = \langle E\rangle/\langle\rho\rangle,$$

then

$$\begin{aligned}
G_{\varepsilon 1} &= \langle E\rangle\partial_x G \equiv \langle E\rangle G_x(x, t; x_1 t_1),\\
G_{p1} &= \langle p\rangle\partial_t G \equiv \langle p\rangle G_t(x, t; x_1, t_1),\\
G_{\varepsilon 1} &= \partial_x G \equiv G_x(x, t; x_1, t_1),\\
G_{v1} &= \partial_t G \equiv G_t(x, t; x_1, t_1).
\end{aligned} \tag{13}$$

Also, we may express the second and third columns in terms of the first by

$$G_{\nu 2} = -G_{\nu 1}(x, t; x_1, t_1)\partial x_1,$$

and (14)

$$G_{\nu 3} = G_{\nu 1}(x, t; x_1 t_1)\partial_{t_1}.$$

where ν denotes, in turn, τ, p, ε and v. Note that the second and third columns are differential operators. The fourth column is the same as the first. Using this Green's function matrix, the inverse of $\langle L\rangle$, i.e. $\langle L\rangle^{-1}\varphi$, where φ is a column matrix of elements $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, is given by

$$\int_{x_1 t_1}\begin{pmatrix} G_{\tau 1} & G_{\tau 2} & G_{\tau 3} & G_{\tau 4}\\ G_{p1} & G_{p2} & G_{p3} & G_{p4}\\ G_{\varepsilon 1} & G_{\varepsilon 2} & G_{\varepsilon 3} & G_{\varepsilon 4}\\ G_{v1} & G_{v2} & G_{v3} & G_{v4}\end{pmatrix}\begin{Bmatrix} \varphi(x_1, t_1)\\ \varphi_2(x_2, t_1)\\ \varphi_3(x_1, t_1)\\ \varphi_4(x_1, t_1)\end{Bmatrix} \mathrm{d}x_1\,\mathrm{d}t_1, \tag{15}$$

where the integration is over the limits of the rod and over all past time. We shall, henceforth, suppress the integral symbol with the understanding that is always to be applied when one encounters a $G_{\nu i}$.

We note that for the infinite statistically homogeneous rod with zero initial conditions and the usual radiation condition we may write an explicit expression for $G(x, t; x_1, t_1)$. (See Ref. [17].)

$$G(x, t; x_1, t_1) = -\frac{c}{2\langle E\rangle}H[c(t-t_1)-|x-x_1|]. \tag{16}$$

Here, $H(x)$ is the Heaviside function. Differentiating equation (16) we write

$$G_x(x, t; x_1, t_1) = \frac{c}{2\langle E\rangle}\,\mathrm{sgn}(x-x_1)\delta[c(t-t_1)-|x-x_1|],$$

and

$$G_t(x_1 t; x_1, t_1) = -\frac{c^2}{2\langle E\rangle}\delta[c(t-t_1)-|x-x_1|],$$

where sgn(x) is the signum function, i.e.

$$\begin{aligned} \operatorname{sgn}(x) &= \;\;1 \qquad x>0 \\ &= -1 \qquad x<0 \end{aligned} \tag{17}$$

and δ is, again, the Dirac function. The reader is referred to Ref. [17] for a clear exposition of the justification and use of symbolic functions.

By direct substitution and matrix manipulation, we can now form the mass operator that appears in the Dyson equation as it applies to the problem at hand. The result is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_{\varepsilon\varepsilon} & A_{\varepsilon v} \\ 0 & 0 & A_{v\varepsilon} & A_{vv} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

where the A's appear as integro-differential operators which we know in the form of infinite series. We write the first few terms of these series:

$$\begin{aligned} A_{\varepsilon\varepsilon} &= \langle E'G_{\varepsilon 2}E'\rangle + \langle E'G_{\varepsilon 2}E'G_{\varepsilon 2}E'\rangle + \langle E'G_{\varepsilon 3}\rho'G_{v2}E'\rangle + \ldots \\ A_{\varepsilon v} &= \langle E'G_{\varepsilon 3}\rho'\rangle + \langle E'G_{\varepsilon 2}E'G_{\varepsilon 3}\rho'\rangle + \langle E'G_{\varepsilon 3}\rho'G_{v3}\rho'\rangle + \ldots \\ A_{v\varepsilon} &= \langle \rho'G_{v2}E'\rangle + \langle \rho'G_{v2}E'G_{\varepsilon 2}E'\rangle + \langle \rho'G_{v3}\rho'G_{v2}E'\rangle + \ldots \\ A_{vv} &= \langle \rho'G_{v3}\rho'\rangle + \langle \rho'G_{v2}E'G_{\varepsilon 3}\rho'\rangle + \langle \rho'G_{v3}\rho'G_{v3}\rho'\rangle + \ldots . \end{aligned} \tag{19}$$

For the infinite statistically homogeneous rod we have explicit expressions for the Green's functions [equations (17)]. We now use these in the above definitions of the A's. After performing some partial integrations we may write the following expressions for the results of the A's acting upon the terms on which they will act in the Dyson equation.

$$\begin{aligned} A_{\varepsilon\varepsilon}\langle\varepsilon\rangle &= \mathscr{E}_1\langle\varepsilon(x,t)\rangle + \int_{-\infty}^{\infty} \mathscr{E}_2(x,x_1)[\partial_{\tau_1}\langle\varepsilon(x_1,t_1)\rangle]\,\mathrm{d}x_1 \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{E}_3(x,x_1,x_2)[\partial_{\tau_1}\partial_{\tau_2}\langle\varepsilon(x_2,t_2)\rangle]\,\mathrm{d}x_2\,\mathrm{d}x_1 \;+ \ldots, \\ A_{\varepsilon v}\langle v\rangle &= \int_{-\infty}^{\infty} \mathscr{C}_{12}(x,x_1)[\partial_{\tau_1}\langle v(x_1,t_1)\rangle]\,\mathrm{d}x_1 \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{C}_{13}(x,x_1,x_2)[\partial_{\tau}\partial_{\tau_2}\langle v(x_2,t_2)\rangle]\,\mathrm{d}x_2\,\mathrm{d}x_1 + \ldots, \\ A_{v\varepsilon}\langle\varepsilon\rangle &= \int_{-\infty}^{\infty} \mathscr{C}_{22}(x,x_1)[\partial_{\tau_1}\langle\varepsilon(x_1,t_1)\rangle]\,\mathrm{d}x_1 \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{C}_{23}(x,x_1)[\partial_{\tau_1}\partial_{\tau_2}\langle\varepsilon(x_2,t_2)\rangle]\,\mathrm{d}x_2\,\mathrm{d}x_1 + \ldots, \end{aligned} \tag{20}$$

$$A_{vv}\langle v\rangle = \int_{-\infty}^{\infty} \mathscr{M}_2(x, x_1)[\partial_{\tau_1}\langle v(x_1, t_1)\rangle]\,dx_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathscr{M}_3(x, x_1, x_2)[\partial_{\tau_1}\partial_{\tau_2}\langle v(x_2, t_2)\rangle]\,dx_2\,dx_1 + \ldots,$$

where

$$\begin{aligned}
\mathscr{E}_1 &= -\langle E\rangle\left[\frac{\langle E'^2\rangle}{\langle E\rangle^2} - \frac{\langle E'^3\rangle}{\langle E\rangle^3} + \ldots\right],\\
\mathscr{E}_2(x, x_1) &= \frac{\langle E\rangle}{2C}\left[\frac{\langle E'(x)E'(x_1)\rangle}{\langle E\rangle^2} - \frac{\langle E'(x)E'^2(x_1)\rangle + \langle E'(x_1)E'^2(x)\rangle}{\langle E\rangle^3} + \ldots\right],\\
\mathscr{E}_3(x, x_1, x_2) &= \frac{\langle \rho\rangle}{4}\left[\frac{\langle E'(x)E'(x_1)E'(x_2)\rangle}{\langle E\rangle^3}\right.\\
&\quad \left. - \operatorname{sgn}(x - x_1)\operatorname{sgn}(x_1 - x_2)\frac{\langle E'(x)\rho'(x_1)\rho'(x_2)\rangle}{\langle E\rangle\langle \rho\rangle^2} + \ldots\right],\\
\mathscr{C}_{12}(x, x_1) &= \frac{\langle \rho\rangle}{2}\left[\operatorname{sgn}(x - x_1)\left(\frac{\langle E'(x)\rho'(x_1)\rangle}{\langle E\rangle\langle \rho\rangle} - \frac{\langle E'^2(x)\rho'(x_1)\rangle}{\langle E\rangle^2\langle \rho\rangle} + \ldots\right)\right],\\
\mathscr{C}_{13}(x, x_1, x_2) &= \frac{\langle \rho\rangle}{4C}\left[\operatorname{sgn}(x_1 - x_2)\frac{\langle E'(x)E'(x_1)\rho'(x_2)\rangle}{\langle E\rangle^2\langle \rho\rangle}\right.\\
&\quad \left. - \operatorname{sgn}(x - x_1)\frac{\langle E'(x)\rho'(x_1)\rho'(x_2)\rangle}{\langle E\rangle\langle \rho\rangle^2} + \ldots\right],\\
\mathscr{C}_{22}(x, x_1) &= \mathscr{C}_{12}(x_1, x),\\
\mathscr{C}_{23}(x, x_1, x_2) &= \mathscr{C}_{13}(x_2, x_1, x),\\
\mathscr{M}_2(x, x_1) &= -\frac{\langle \rho\rangle}{2C}\left[\frac{\langle \rho'(x)\rho'(x_1)\rangle}{\langle \rho\rangle^2} + \ldots\right],\\
\mathscr{M}_3(x, x_1, x_2) &= -\frac{\langle \rho\rangle}{4C^2}\left[\operatorname{sgn}(x - x_1)\operatorname{sgn}(x_1 - x_2)\frac{\langle \rho'(x)E'(x_1)\rho'(x_2)\rangle}{\langle \rho\rangle^2\langle E\rangle}\right.\\
&\quad \left. - \frac{\langle \rho'(x)\rho'(x_1)\rho'(x_2)\rangle}{\langle \rho\rangle^3} + \ldots\right].
\end{aligned} \tag{21}$$

In these equations a series of dots indicates an infinite number of terms and

$$\partial_{\tau_i}\varphi(t_i) \equiv [\partial_{t_i}\varphi(t_i)] \qquad t_i = t_{i-1} - \frac{|x_{i-1} - x_i|}{C}. \tag{22}$$

We note that the representation of the A's as given by equations (20) is not unique. We elected to use the Dirac function to eliminate the integration over t instead of eliminating the integration over x. Also, we have the equality $\partial_t\langle \varepsilon\rangle = \partial_x\langle v\rangle$ which could be introduced.

Dropping the matrix notation, the equations governing the mean field quantities in a stochastic linearly elastic rod are:

$$\partial_x\langle\tau\rangle = \partial_t\langle p\rangle, \tag{22a}$$

$$\langle\tau\rangle = [\langle E\rangle + A_{\varepsilon\varepsilon}]\langle\varepsilon\rangle + A_{\varepsilon v}\langle v\rangle, \tag{22b}$$

$$\langle p\rangle = A_{v\varepsilon}\langle\varepsilon\rangle + [\langle\rho\rangle + A_{vv}]\langle v\rangle, \tag{22c}$$

$$\langle\varepsilon\rangle = \partial_x\langle u\rangle, \tag{22d}$$

$$\langle v\rangle = \partial_t\langle u\rangle. \tag{22e}$$

A comparison might be made between these equations and those that govern the field quantities in a deterministic linearly elastic rod. Such a comparison can lead to our viewing the former as a generalization of the latter. In the generalized equations the stress field is obtained by a linear combination of operations on the strain and velocity fields, as is the linear momentum field. In the specialized equations the stress field is given by an algebraic operation on the strain field alone and the linear momentum field by an algebraic operation on the velocity field alone. Too much significance should not be attached to this formal identification, however. More confusion than understanding will arise from attempting to view the above equations as representing those that govern a generalized continuum.

In the next sections of the paper we shall restrict attention to the infinite statistically homogeneous rod in discussing the usefulness of the formulism developed and in investigating the dispersion spectrum. In the final section we shall consider what changes might be expected for the finite rod. Before proceeding, however, we draw the reader's attention once again to the discussion in the introduction on when we can extract some information of the response of an individual rod from knowledge of the average response of the ensemble.

DISCUSSION OF GENERAL FORMULATION—EFFECTIVE MODULUS THEORY

While equations (22) with the A operators given by equations (20) and (21) do define a formulation for predicting the mean field response, this formulation is not very useful as it stands. This is because the definitions of the A operators contain infinite series involving the joint statistical moments of the stochastic functions E' and ρ'. We can really only use these definitions in a direct fashion if we can either truncate these series or else sum them.

A truncation can be introduced in the case in which the random variations are small relative to the mean values, i.e.

$$\frac{\langle E'^m\rho'^n\rangle}{\langle E\rangle^m\langle\rho\rangle^n} \ll 1. \tag{23}$$

In such a case we can obtain a consistent approximation of our general formulation that is good to second order in the random variations. Equations (22b) and (22c) become

$$\begin{aligned}\langle\tau(x,t)\rangle = \langle E\rangle\left[1-\frac{\langle E'^2\rangle}{\langle E\rangle^2}\right]\langle\varepsilon(x,t)\rangle + \frac{\langle E\rangle}{2C}\int_{-\infty}^{\infty}\frac{\langle E'(x)E'(x_1)\rangle}{\langle E\rangle^2}[\partial_{\tau_1}\langle\varepsilon(x_1,t_1)\rangle]\,dx_1\\ +\frac{\langle\rho\rangle}{2}\int_{x_1}\operatorname{sgn}(x-x_1)\frac{\langle E'(x)\rho'(x_1)\rangle}{\langle E\rangle\langle\rho\rangle}[\partial_{\tau_1}\langle v(x_1,t_1)\rangle]\,dx_1\end{aligned} \tag{24b}$$

$$\langle p(x,t)\rangle = -\frac{\langle \rho\rangle}{2}\int_{x_1}\operatorname{sgn}(x-x_1)\frac{\langle \rho'(x)E'(x_1)\rangle}{\langle \rho\rangle\langle E\rangle}[\partial_{\tau_1}\langle \varepsilon(x_1,t_1)\rangle]\,dx_1 + \langle \rho\rangle\langle v(x,t)\rangle - \frac{\langle \rho\rangle}{2C}\int_{x_1}\frac{\langle \rho'(x)\rho'(x_1)\rangle}{\langle \rho\rangle^2}[\partial_{\tau_1}\langle v(x_1,t_1)\rangle]\,dx_1. \tag{24c}$$

With these specific forms the formulation developed becomes useful. The only statistical information, in addition to the average values of E and ρ, that is needed is contained in the two point moments, $\langle E'(x)E'(x_1)\rangle$, $\langle \rho'(x)E'(x_1)\rangle$ and $\langle \rho'(x)\rho'(x_1)\rangle$. The measurement of these quantities is conceptually not difficult. It is to be noted that in the case of homogeneous statistics and an infinite rod the integrals appearing in equations (24b) and (24c) are of the convolution type suggesting the desirability of a Fourier representation, that is, cast the formulation into k space. In the next section we shall investigate the dispersion spectrum for this weakly inhomogeneous case.

Except for this weakly inhomogeneous case the author does not evision a practically meaningful way in which he can either truncate or sum the series appearing in the definitions of the A operators. Therefore, if the formulation is to be useful for other than the weakly inhomogeneous rod it is necessary to look to other type approximations in order to simplify it. One obvious approximation would be valid in those cases in which the length scale on which the random variations in E' and ρ' are observable is different from the length scale on which the variations in the mean field quantities are observable. If we introduce l to denote the largest correlation distance defined by the variations in E' and ρ' and L and L/c to denote the smallest length and time variations of the mean field quantities, then we can look to the form of the equations in the limit of l/L becoming vanishingly small. To obtain this form we expand $\langle \varepsilon\rangle$ and $\langle v\rangle$ as they appear in the integrals of equations (20) as Taylor series about the point (x, t). This allows us to write, for example,

$$A_{\varepsilon\varepsilon}\langle \varepsilon\rangle = \mathscr{E}_1\langle \varepsilon(x,t)\rangle + \bar{\mathscr{E}}_2 l[\partial_t\langle \varepsilon(x,t)\rangle] + \bar{\mathscr{E}}_{31} l^2[\partial_t\partial_t\langle \varepsilon(x,t)\rangle] + \bar{\mathscr{E}}_{32} l^2[\partial_x\partial_t\langle \varepsilon(x,t)\rangle] + \ldots,$$

where

$$\bar{\mathscr{E}}_2 l = \int_{x_1}\mathscr{E}_2(x,x_1)\,dx_1$$

$$\bar{\mathscr{E}}_{31} l^2 = \int_{x_1}\int_{x_2}\mathscr{E}_3(x,x_1,x_2)\,dx_1\,dx_2 - \frac{1}{C}\int_{x_1}|x-x_1|\mathscr{E}_2(x,x_1)\,dx_1 \tag{25}$$

$$\bar{\mathscr{E}}_{32} l^2 = \int_{x_1}(x_1-x)\mathscr{E}_2(x,x_1)\,dx_1$$

etc. The parameters, $\bar{\mathscr{E}}$, are all of order unity. Similar results are obtained by considering the other expressions in equations (20). Referring to the definition of L, we see that these series contain terms of increasingly higher order of (l/L). We might truncate these series retaining only terms of order $(l/L)^0$ to arrive at a zeroth order approximation. The result is that we may express equations (23b) and (23c) as

$$\langle \tau\rangle = [\langle E\rangle - \mathscr{E}_1]\langle \varepsilon\rangle \tag{26b}$$

$$\langle p\rangle = \langle \rho\rangle\langle v\rangle. \tag{26c}$$

We let

$$E_{\text{eff}} \equiv [\langle E\rangle - \mathscr{E}_1] = \frac{\langle E\rangle\langle 1/E\rangle + \langle E'/E\rangle}{\langle 1/E\rangle}. \tag{27}$$

Similarly, we might obtain a first order approximation by retaining all terms of order (l/L) that appear in these series while rejecting all terms of higher order. The result is

$$\langle \tau\rangle = E_{\text{eff}}\langle \varepsilon\rangle + \bar{\mathscr{E}}_2 l[\partial_t\langle \varepsilon\rangle] + \bar{\mathscr{C}}_{12} l[\partial_t\langle v\rangle], \tag{28b}$$

$$\langle p\rangle = \langle \rho\rangle\langle v\rangle + \bar{\mathscr{M}}_2 l[\partial_t\langle v\rangle] + \bar{\mathscr{C}}_{21} l[\partial_t\langle \varepsilon\rangle], \tag{28c}$$

where $\bar{\mathscr{E}}_2$ is given by equations (25) and $\bar{\mathscr{C}}_{12}$, $\bar{\mathscr{C}}_{21}$ and $\bar{\mathscr{M}}_2$ have similar definitions replacing $\mathscr{E}_2$ in equation (25) by $\mathscr{C}_{12}$, $\mathscr{C}_{21}$ and $\mathscr{M}_2$, respectively. We note that in the case of homogeneous statistics $\bar{\mathscr{C}}_{12} = \bar{\mathscr{C}}_{21} = 0$.

The zeroth order approximation is recognized as an "effective modulus" theory. We notice that for the problem being investigated it is possible to recognize the value of the infinite series that defines the effective modulus. This is an accident of the specific case at hand. In general we cannot expect to be so fortunate. In applying the effective modulus, low frequency, long wavelength approximation we shall ignore the analytical prescription developed for the effective modulus. Rather, we view the effective modulus theory as a phenomenological theory in which the effective modulus enters as a parameter to be determined by comparing the predictions of the theory with measured results obtained in test situations.

We can make the following statements of the effective modulus theory. The solutions that it predicts must satisfy the conditions on which it was derived (i.e. $l/L \ll 1$) if these solutions are to be valid. The conditions for the validity of the effective modulus theory are the same as those justifying the invoking of an ergodic type hypothesis. Thirdly, it is not difficult to show that the effective modulus obtained from this dynamic analysis agrees with that one obtains on ignoring inertia effects from the outset. It might be noted that the statical problem is simple enough that one can carry out the derivation procedure without resorting to any iteration scheme. The reader is referred to Ref. [18] to see how this is accomplished. Fourth, it is noticed that only one point information in the variations in E' enters the definition of E_{eff}. Finally, the average mass and not an effective mass enters the effective modulus theory. As a general rule for problems of this type, it is to be expected that the last two observations are peculiar to the specific problem at hand. The effective moduli one obtains in considering three-dimensional linearly elastic solids in the absence of inertia effects, for example, are known to depend on three and more point information. It would be most interesting if only the average mass and not an effective mass entered the theory as a general rule. The average mass is the quantity that one would obtain by simply weighing the solid.

The following comments can be made of the first order approximation to the general formulation. Firstly, the formulation is not uniquely expressed by equations (28b) and (28c). For example, we might make use of the equality $\partial_t\langle \varepsilon\rangle = \partial_x\langle v\rangle$ to obtain an identical although seemingly different expression. Secondly, it can be expected to admit solutions which may be viewed as modifications of the solutions admitted by the zeroth order approximation in that the difference between the two sets of solutions will vanish in the limit of l/L vanishing. The term $\bar{\mathscr{E}}_2 l\, \partial_t\langle \varepsilon\rangle$ appearing in equation (28b) is a dissipative type term. Hence, we expect that the first order equations will predict a response that is like that for a solid in which we allow some energy dissipation. We consider this further in the next section. Since the order of the system of equations of the first order approximation is higher than that of the zeroth order approximation it will admit solutions that have no counterpart in the latter theory. These additional solutions will be of questionable validity, however, since they will probably correspond to a situation in which we are not justified in truncating terms of order (l/L) or higher.

The effects of relaxing the low frequency, long wavelength assumption will be illuminated in the next section when we consider the dispersion spectrum for the weakly inhomogeneous rod. Before proceeding to these specific calculations, however, it is worthwhile to make one final comment of the general formulation and that is that a series of integration by parts and algebraic manipulations allows the writing of equations (22b) and (22c) as

$$\langle \tau(x,t)\rangle = \int_{x_1,t_1} \mathscr{E}(x,t;x_1,t_1)\langle \varepsilon(x_1,t_1)\rangle \,\mathrm{d}x_1\,\mathrm{d}t_1 + \int_{x_1,t_1} \mathscr{C}(x,t;x_1,t_1)\partial_{t_1}\langle v(x_1,t_1)\rangle \,\mathrm{d}x_1\,\mathrm{d}t_1 \tag{29b}$$

and

$$\langle p(x,t)\rangle = \int_{x_1,t_1} \mathscr{C}(x_1,t_1;x,t)\langle \varepsilon(x_1,t_1)\rangle \,\mathrm{d}x_1\,\mathrm{d}t_1 + \int_{x_1,t_1} \mathscr{M}(x,t;x_1,t_1)\partial_{t_1}\langle v(x_1,t_1)\rangle \,\mathrm{d}x_1\,\mathrm{d}t_1 + \langle \rho\rangle\langle v(x,t)\rangle. \tag{29c}$$

We may ignore the analytical prescription for the four two-point functions and treat them as phenomenological parameters to be determined by matching prediction with experiment. Of course, there is an enormous amount of freedom in these four functions; far too much for us to hope to be able to uniquely determine them in test situations. A more realistic approach is to accept the general formulation with equations (29b) and (29c) as a foundation for developing an approximate formulation. A given approximation will be defined by restricting the functional form of $\mathscr{E}$, $\mathscr{C}$ and $\mathscr{M}$. We do not pursue this possibility any further in the present paper, but refer the reader to [19] for a discussion of this possibility as it arises in a different context.

DISPERSION SPECTRUM FOR INFINITE WEAKLY INHOMOGENEOUS RODS

In this section we consider the reduced formulation obtained upon making the weakly inhomogeneous approximation. We are interested in obtaining the infinite rod dispersion spectrum. Accordingly, we let

$$\varphi(x,t) = \hat{\varphi}(k,\omega)\exp[i(kx-\omega t)] \tag{30}$$

where $\phi(x, t)$ denotes, in turn, $\langle\tau\rangle$, $\langle p\rangle$, $\langle\varepsilon\rangle$, $\langle v\rangle$ and $\langle u\rangle$. Substitution in equations (22a), (24b), (24c), (22d) and (22e) and factoring the common exponential term gives

$$
\begin{aligned}
ik\langle\hat{\tau}\rangle &= -i\omega\langle\hat{p}\rangle,\\
\langle\hat{\tau}\rangle &= [E_{\mathrm{eff}} - i\eta^2\omega^2\langle\rho\rangle\eta_{EE}(k,\omega)]\langle\hat{\varepsilon}\rangle - \eta^2\omega^2\langle\rho\rangle c\eta_{E\rho}(k,\omega)\langle\hat{v}\rangle,\\
\langle\hat{p}\rangle &= \eta^2\omega^2\langle\rho\rangle c\eta_{E\rho}(k,\omega)\langle\hat{\varepsilon}\rangle + \left[\langle\rho\rangle + i\eta^2\omega^2\frac{\langle\rho\rangle^2}{\langle E\rangle}\eta_{\rho\rho}(k,\omega)\right]\langle\hat{v}\rangle,\\
\langle\hat{\varepsilon}\rangle &= ik\langle\hat{u}\rangle,\\
\langle\hat{v}\rangle &= -i\omega\langle u\rangle.
\end{aligned}
\tag{31}
$$

In these equations

$$E_{\mathrm{eff}} = \langle E\rangle\left[1 - \frac{\langle E'^2\rangle}{\langle E\rangle^2}\right],$$

$$\eta_{EE}(k,\omega) = \int_0^\infty N_{EE}(k_0^{-1}r)\cos(kk_0^{-1}r)\exp(ir)\,\mathrm{d}r,$$

$$\eta_{E\rho}(k,\omega) = \int_0^\infty N_{E\rho}(k_0^{-1}r)\sin(kk_0^{-1}r)\exp(ir)\,\mathrm{d}r,$$

and

$$\eta_{\rho\rho}(k,\omega) = \int_0^\infty N_{\rho\rho}(k_0^{-1}r)\cos(kk_0^{-1}r)\exp(ir)\,\mathrm{d}r, \tag{32}$$

where

$$k_0 = \omega/c \equiv \omega\sqrt{(\langle\rho\rangle/\langle E\rangle)},$$

$$\eta^2 N_{EE}(|x - x_1|) = \langle E'(x)E'(x_1)\rangle/\langle E\rangle^2,$$

$$\eta^2 N_{E\rho}(|x - x_1|) = \langle E'(x)\rho'(x_1)\rangle/\langle E\rangle\langle\rho\rangle = \eta^2 N_{\rho E}(|x - x_1|),$$

and

$$\eta^2 N_{\rho\rho}(|x - x_1|) = \langle\rho'(x)\rho'(x_1)\rangle/\langle\rho\rangle^2.$$

The parameter η^2 is a scaling factor introduced to allow the N's to be of order unity. In the weakly inhomogeneous medium,

$$\eta^2 \ll 1.$$

The quantity k_0 is recognized as the wave number at frequency ω in the "mean" rod. The fact that the correlation functions depend only on the difference coordinate is a result of the assumed statistical homogeneity. By substitution and rearranging we now obtain the following characteristic equation,

$$\left(\frac{\bar{k}^2}{\bar{k}_{\mathrm{eff}}^2} - 1\right) + \eta^2[\bar{k}\eta_{\rho E}(\bar{k}, k_0) - i\bar{k}^2\eta_{EE}(\bar{k}, k_0) - i\eta_{\rho\rho}(\bar{k}, k_0)] = 0. \tag{33}$$

Here,

$$\bar{k} = k/k_0$$
$$\bar{k}_{\text{eff}} = \omega\sqrt{(\langle\rho\rangle/E_{\text{eff}})}/k_0 \tag{34}$$

denote the wave numbers of our small perturbation formulation, and of a small perturbation, effective modulus formulation both non-dimensionalized using the "mean" rod wave number. The roots of this characteristic equation, denoted by

$$\bar{k}_i; \qquad i = 1, \ldots, n,$$

where n is however many roots the equation happens to have for a given k_0, give the branches of the desired dispersion spectrum. In a general situation we should expect that the η's will be transcendental functions thereby giving rise to a dispersion spectrum with an infinite number of branches. In special cases, however, the correlation functions may be such as to result in η's that are algebraic functions. In such a case the dispersion spectrum will have only a finite number of branches.

To solve the characteristic equation we can make use of the fact that $\eta^2 \ll 1$ for the equation to be valid. Thus the second term will offer a negligible to the equation provided a root is not so located as to make the bracketed quantity very large. We obtain approximate values of the desired roots that is good to order η^2 by a perturbation procedure. Our base roots are the roots of the characteristic equation for $\eta = 0$ and the values of $\bar{k}$ for which the bracketed term becomes singular. For $\eta = 0$, the characteristic equation reduces to that of the homogeneous average rod and hence we obtain for a base root $\bar{k} = 1$. Perturbing this value and noting that

$$k_{\text{eff}}^2 = 1 + \eta^2 N_{EE}(0) + O(\eta^4),$$

we obtain the following approximation of a root of the secular equation that is good to order η^2.

$$\bar{k}_1^2 = 1 + \eta^2[N_{EE}(0) - i\eta_{EE}(1, k_0) + \eta_{\rho E}(1, k_0) - i\eta_{\rho\rho}(1, k_0)]. \tag{35}$$

In obtaining equation (35) we have made no explicit assumptions of the relative sizes of l, the largest correlation length, and $L = k_0^{-1}$, a characteristic length associated with the variations in the mean field. If we now assume that l/L is small we can obtain simplified expressions for the η's. This is done by noting that in the integral's the N's are non-zero only for r small, hence we are justified in expanding the trigonometric functions as power series and in truncating. The result is

$$\bar{k}_1^2 = 1 + \eta^2\{N_{EE}(0) + k_0^2 l^2(\sigma_{EE}^{(2)} + \sigma_{\rho\rho}^{(2)}) - i[k_0 l(\sigma_{EE}^{(1)} + \sigma_{\rho\rho}^{(1)}) - k_0^2 l^2 \sigma_{\rho E}^{(2)}] + O(k_0^3 l^3)\}, \tag{36}$$

where

$$\sigma_{EE}^{(n)} = \frac{1}{l^n}\int_0^\infty r^{n-1} N_{EE}(r)\,\mathrm{d}r,$$

with a similar definition for $\sigma_{\rho\rho}^{(n)}$ and $\sigma_{\rho E}^{(n)}$. Keeping only terms of order $(k_0 l)^0$, the above result agrees with the small perturbation effective modulus theory. Retaining terms of order $(k_0 l)$, the above result agrees with the small perturbation first order approximation. To this order we see that this branch exhibits spatial decay. Retaining terms of order $(k_0 l)^2$ predicts a branch that exhibits both dispersion and spatial decay. These results are in agreement with those of earlier researchers such as Karal and Keller [13] who investigated waves in a three-dimensional elastic solid.

To obtain approximate expressions for the other branches we perturb those values of $\bar{k}$ for which the bracketed term in equation (33) becomes singular. This will occur for those values of $\bar{k}$ for which the η's become singular. If we consider, for example, $\eta_{EE}(k, k_0)$, we see that we can express it as

$$\eta_{EE}(k_1 k_0) = \tfrac{1}{2}[\hat{N}_{EE}(k_0+k)+\hat{N}_{EE}(k_0-k)]$$

where

$$\hat{N}(s) = \int_0^\infty N(x)\exp(isx)\,\mathrm{d}x$$

that is, $\hat{N}(\rho)$ is a reciprocal space representation of the correlation function. Thus $\eta_{EE}(k, k_0)$ becomes singular for those values of $\bar{k}$ for which $\hat{N}[(1+\bar{k})k_0]$ or $\hat{N}[(1-\bar{k})k_0]$ becomes singular. For a correlation function that falls to zero in a distance l, it is not difficult to show that $\hat{N}[(1+\bar{k})k_0]$ and $\hat{N}[(1-\bar{k})k_0]$ will become singular only for a value of $\bar{k}$ such that $\mathrm{Im}(\bar{k}k_0)$ is of order l^{-1}. Thus except for the branch that is a small perturbation of the single non-dispersive, non-decaying branch of the homogeneous average rod, all branches of the infinite rod dispersion spectrum define waves that decay in a distance that is of the same order as the correlation length, l. This is in agreement with a similar result obtained by Tatarski and Gertenshtein [20]. These authors investigated the stochastic Helmholtz operator for an assumed form for the correlation function.

The physical significance to attach to the infinite rod dispersion spectrum is easily understood by considering a point source in an infinite rod which is capable of projecting energy in a single direction. If the source is a Helmholtz resonator of frequency ω, then the waves defined by $k_i(\omega)$ combine to give the steady state response of the rod. The amplitude and phase to assign to the individual waves depend on the details of the resonator. We have found that outside of a region of size that is of the order of magnitude of a correlation length we need consider only one wave and this wave is a small perturbation of that which occurs in the homogeneous "mean" rod. We note that even this decays with the decay distance for long wavelengths being given by

$$\lambda_0^2/\eta^2 l$$

where λ_0 is the length of the wave in the homogeneous average rod, l is the correlation length of the variations of E' and η gives the size of the variations in E' relative to $\langle E \rangle$. It should be noted that the fact that no response is ever noticed a finite distance from the source is not a consequence of energy dissipation. Rather, it results from the energy eventually being transformed into totally "incoherent" energy.

The presence of the additional branches, all of which are strongly evanescent introduces the possibility of a situation in which a portion of the energy will be in a coherent form only within a very limited region of the source. The investigation of this possibility is left for a future study.

CONCLUDING REMARKS

In this paper we have derived a formulation that must be satisfied by the mean field quantities for a statistical assemblage of linearly elastic rods. This formulation is given by equations (22). In general the A operators which determine the averaged stress and linear momentum fields in terms of the averaged strain and velocity fields depend on the joint statistics of $E(x)$ and $\rho(x)$ and of the boundary conditions that are to be satisfied by each element of the assemblage. Thus, the amount of information contained in the A's is enormous; probably, far too much for us to ascertain in a meaningful problem. We cannot, therefore, consider this general formulation to be of much immediate use in a practical situation. We hasten to point out, however, that this difficulty is an inherent difficulty of the problem and is not a fault of the formulation derived. In the general situation, the averaged response of the assemblage does depend on the complete joint statistics of $E(x)$ and $\rho(x)$ and a formulation that requires less than this information can be at best an approximation. It is also to be noted that the general formulation can serve as a foundation for deriving approximations in those situations in which we cannot put it to immediate use. This point is considered in detail in a different context in Ref. [19].

There are two physically important situations in which the general formulation reduces to formulations which can be of immediate use. One is the situation in which the variations in $E(x)$ and $\rho(x)$ are vanishingly small and one is the situation in which the length scale on which these variations are observable is different from the length scale on which the variations in the averaged field quantities are observable. The former occurrence results in a simplified set of equations that contains only two point moments of the variations in $E(x)$ and $\rho(x)$. The latter occurrence results in an "effective modulus" theory as a first approximation. With respect to the weakly inhomogeneous assemblage, an attempt could be made to determine its averaged response using a classical perturbation procedure. The result of this approach is often termed the Born approximation. The Born approximation is marred by the fact that it is not a uniformly valid approximation. It is valid in the weakly inhomogeneous assemblage only over short propagation distances. This fact is now well understood and is discussed in Ref. [16]. With respect to the two length scale situation, it is just this occurrence which allows one to invoke an ergodic hypothesis and think in terms of spatial averages taken over the inner length scale. Both of these physically important approximations have been considered in some detail for the statistically homogeneous assemblage of infinite rods.

Introducing a boundary to the rods introduces many complications that are to be left for future studies. We can speculate on one result of introducing such a boundary by considering a special case in which we have semi-infinite weakly inhomogeneous rods bonded to semi-infinite homogeneous rods. The Young's modulus and the mass density of the homogeneous rods are to be equal to the averaged Young's modulus and the averaged mass density of the stochastic rods. In such a situation the Green's functions to use in the definitions of the A operators are the same as in the infinite stochastic rods, the only difference in the definitions of the A operators being the range of the integrations contained therein. For such a situation one can solve a specific problem of a wave train generated at infinity in the homogeneous rod, which then propagates toward and interacts with the boundary between the homogeneous rod and the inhomogeneous rod to which it is bonded. The solution procedure requires an application of the Weiner–Hopf technique. The result shows that all the waves of the infinite rod dispersion spectrum will be generated at the boundary. This is true no matter the relative lengths of the wave in the homogeneous rod and the variations in $E(x)$ and $\rho(x)$. An "effective modulus" theory was seen to contain only one of these waves. It was also seen that all of the waves not contained in the effective modulus theory were rapidly attenuated (spatially). From this one can conclude that in a bounded rod forced in such a manner as to satisfy a two length scale requirement the "effective modulus" theory is a valid interior theory. One must introduce, however, a boundary layer theory that governs the response between this interior region and an actual physical boundary. Some aspects of this feature are contained in Kupiec *et al.* [21].

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Абстракт—Исследуется динамическое поведение линейного упругого стержня, которого свойства материала описываются случайными функциями положения вдолв стержня. Определяется общая формулировка, которая удовлетворяет множеству усредненных свойств поля. Рассматривается она в некоторых подробностях, для таких случаев, в которых значения случайных функций стремятся к нулю, а также для таких же, в которых длина масштаба наблюдаемых измерений является малой по сравнению с длиной масштаба, для которой можно наблюдатъ измерение усредненных свойств поля. Для предыдущего случая получается спектр рассеяния для бесконечного стержня. Для следующего же даётся теория динамического ,,эффективного модуля". Обсуждается физические выводы результатов.